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A Study of Higher Order Terms in Shallow Water Waves via Modified PLK Method

Abstract: In this work, by utilizing the modified PLK (Poincare-Lighthill-Kou) method, we studied the propagation of weakly nonlinear waves in a shallow water theory and obtained the Korteweg-deVries (KdV) and the linearized KdV equations with non-homogeneous term as the governing equations of various order terms in the perturbation expansion. The result obtained here is exactly the same with that of Kodama and Taniuti [6], who employed the so-called “re-normalization method”. Seeking a progressive wave solution to these evolution equations we obtained the speed correction terms so as to remove some possible secularities. The result obtained here is consistent with the results of Demiray [12], where in the modified reductive perturbation method had been utilized.

Keywords: shallow water waves, modified PLK method, solitary waves

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1 Introduction

The studies of nonlinear waves of various fields in physics and engineering, by use of the reductive perturbation method in the long-wave approximation, lead to the Korteweg-deVries equation as the evolution equation [1, 2]. The study of the higher order terms in the perturbation expansion by use of the reductive perturbation method leads to some secularities [3, 4]. To remove such secularities Sugimoto and Kakutani [5] introduced additional slow variables both in space and time in reductive perturbation theory, but their result was not supported by other methods. Kodama and Taniuti [6] presented the re-normalization procedure of the velocity of the KdV soliton. As stated in [6], the lowest order term in the perturbation expansion is governed by the conventional KdV equation whereas the higher order terms are governed by the linearized KdV equation with non-homogeneous

term

$$\frac{\partial u_n}{\partial \tau} - 6 \frac{\partial}{\partial \xi} (u_1 u_n) + \frac{\partial^3 u_n}{\partial \xi^3} = S_n(u_1, u_2, \dots, u_{n-1}), \quad (1)$$

where ξ and τ are slow variables in reductive perturbation method, u_1, u_2, \dots, u_n are the coefficient functions of the perturbation expansion and $S_n(u_1, u_2, \dots, u_{n-1})$ is the non-homogeneous term. It is a well-known result that if u_1 is the solution of the conventional KdV equation $u_{1,\xi}$ will be the solution of the homogeneous equation

$$\frac{\partial u_n}{\partial \tau} - 6 \frac{\partial}{\partial \xi} (u_1 u_n) + \frac{\partial^3 u_n}{\partial \xi^3} = 0. \quad (2)$$

If the non-homogeneous term S_n contains a term proportional to $u_{1,\xi}$, the particular solution of Eq. (1) will have a secular part, which contradicts the assumption of uniform convergence of perturbation expansion. In order to remove such a secularity, roughly speaking, Kodama and Taniuti [6] added both sides of the Eq. (1) a term proportional to $u_{1,\xi}$ and determined the proportionality coefficient so as to remove the secularity in the particular integral. They called this method as “the re-normalization method”. This approach had been criticized by several scientists [7, 8, 9] who found this approach somewhat artificial.

In this work, utilizing the modified PLK method [10, 11], which is based on expanding the independent stretched coordinate as well as the dependent variables into a perturbation series, we studied the propagation of weakly nonlinear waves in a shallow water theory and obtained the KdV (nonlinear) and the linearized (degenerate) KdV equations with non-homogeneous term as the governing equations of various order terms in the perturbation expansion. The result obtained here is exactly the same as that of Kodama and Taniuti [6], who employed the so-called “re-normalization method”, which is quite heuristic. By seeking a progressive wave solution to these evolution equations we determined the speed correction terms so as to remove possible secularities that occur in the solution. The result obtained here is consistent with the results of Demiray [12], in which the modified reductive perturbation method had been utilized.

2 Modified PLK formalism for shallow water waves

We consider two dimensional incompressible inviscid fluid in a constant gravitational field g acting in the negative z^* -direction. The space coordinates are denoted by (x^*, z^*) and the corresponding velocity components by (u^*, w^*) . The equations of motion describing such a fluid are:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial w^*}{\partial z^*} = 0 \quad (\text{incompressibility}), \quad (3)$$

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + w^* \frac{\partial u^*}{\partial z^*} + \frac{1}{\rho} \frac{\partial P^*}{\partial x^*} = 0, \quad (4)$$

$$\frac{\partial w^*}{\partial t^*} + u^* \frac{\partial w^*}{\partial x^*} + w^* \frac{\partial w^*}{\partial z^*} + \frac{1}{\rho} \frac{\partial P^*}{\partial z^*} + g = 0, \quad (5)$$

where t^* is the time parameter, ρ is the mass density and P^* is the fluid pressure function. Assuming that the flow is ir-rotational, the velocity components can be expressed in terms of scalar potential ϕ^* as

$$u^* = \frac{\partial \phi^*}{\partial x^*}, \quad w^* = \frac{\partial \phi^*}{\partial z^*}. \quad (6)$$

Then, the incompressibility condition becomes

$$\frac{\partial^2 \phi^*}{\partial x^{*2}} + \frac{\partial^2 \phi^*}{\partial z^{*2}} = 0, \quad (7)$$

and the Euler equation reads

$$\frac{P^* - P_0^*}{\rho} + \frac{\partial \phi^*}{\partial t^*} + \frac{1}{2} \left[\left(\frac{\partial \phi^*}{\partial x^*} \right)^2 + \left(\frac{\partial \phi^*}{\partial z^*} \right)^2 \right] + g z^* = 0, \quad (8)$$

where P_0^* is the atmospheric pressure.

We consider the case of fluid of height h_0 , bounded above by a steady atmospheric pressure P_0^* . Let the upper surface be described by $z^* = \eta^*(x^*, t^*)$. The kinematical boundary condition on this surface can be expressed as:

$$\frac{\partial \phi^*}{\partial z^*} = \frac{\partial \eta^*}{\partial t^*} + \frac{\partial \phi^*}{\partial x^*} \frac{\partial \eta^*}{\partial x^*}, \quad \text{on } z^* = \eta^*. \quad (9)$$

From the equation (8), the dynamical boundary condition on this surface reads

$$\frac{\partial \phi^*}{\partial t^*} + \frac{1}{2} \left[\left(\frac{\partial \phi^*}{\partial x^*} \right)^2 + \left(\frac{\partial \phi^*}{\partial z^*} \right)^2 \right] + g \eta^* = 0, \quad \text{on } z^* = \eta^*. \quad (10)$$

Finally, the lower boundary is supposed to be rigid horizontal plane. Therefore, at $z^* = -h_0$, the normal component of the velocity must vanish, i.e.,

$$\frac{\partial \phi^*}{\partial z^*} = 0, \quad \text{at } z^* = -h_0. \quad (11)$$

At this stage it is convenient to introduce the following non-dimensional quantities:

$$\begin{aligned} x^* &= h_0 x, & z^* &= h_0 z, & t^* &= \frac{h_0}{c_0} t, \\ \phi^* &= c_0 h_0 \hat{\phi}, & \eta^* &= h_0 \hat{\eta}, & P^* &= \rho c_0^2 p, \\ P_0^* &= \rho c_0^2 p_0, & c_0 &= (g h_0)^{1/2}. \end{aligned} \quad (12)$$

Introducing (12) into Eqs. (7)–(11) the following non-dimensional equations are obtained:

$$\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial z^2} = 0, \quad (13)$$

$$\frac{\partial \hat{\phi}}{\partial z} = \frac{\partial \hat{\eta}}{\partial t} + \frac{\partial \hat{\phi}}{\partial x} \frac{\partial \hat{\eta}}{\partial x}, \quad \text{at } z = \hat{\eta}, \quad (14)$$

$$\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2} \left[\left(\frac{\partial \hat{\phi}}{\partial x} \right)^2 + \left(\frac{\partial \hat{\phi}}{\partial z} \right)^2 \right] + \hat{\eta} = 0, \quad \text{at } z = \hat{\eta}, \quad (15)$$

$$\frac{\partial \hat{\phi}}{\partial z} = 0, \quad \text{at } z = -1. \quad (16)$$

Now, we shall consider the propagation of nonlinear waves in shallow water approximation to the above equations by applying the modified PLK method. According to this method the following slow variables in power series form is introduced:

$$\epsilon^{1/2}(x - t) = \xi + \sum_{n=1}^{\infty} \epsilon^n P_n(\tau), \quad \tau = \epsilon^{3/2} t, \quad (17)$$

where ϵ is a small parameter characterizing the smallness of certain physical entities and $P_n(\tau)$ are some unknown functions to be determined from the solution. For our future purposes, we introduce the following new variables:

$$\hat{\phi} = \epsilon^{1/2} \phi, \quad \hat{\eta} = \epsilon \eta. \quad (18)$$

Introducing Eqs. (17) and (18) into Eqs. (13)–(16) we have

$$\frac{\partial^2 \phi}{\partial z^2} + \epsilon \frac{\partial^2 \phi}{\partial \xi^2} = 0, \quad (19)$$

$$\frac{\partial \phi}{\partial z} = -\epsilon \left(1 + \sum_{n=1}^{\infty} \epsilon^{n+1} \frac{dP_n(\tau)}{d\tau} \right) \frac{\partial \eta}{\partial \xi} + \epsilon^2 \frac{\partial \eta}{\partial \tau} + \epsilon^2 \frac{\partial \phi}{\partial \xi} \frac{\partial \eta}{\partial \xi}, \quad \text{at } z = \epsilon \eta, \quad (20)$$

$$-\left(1 + \sum_{n=1}^{\infty} \epsilon^{n+1} \frac{dP_n(\tau)}{d\tau} \right) \frac{\partial \phi}{\partial \xi} + \epsilon \frac{\partial \phi}{\partial \tau} + \frac{1}{2} \left[\epsilon \left(\frac{\partial \phi}{\partial \xi} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right] + \eta = 0, \quad \text{at } z = \epsilon \eta, \quad (21)$$

$$\frac{\partial \phi}{\partial z} = 0, \quad \text{at } z = -1. \quad (22)$$

Now, we expand the functions ϕ and η into a suitable power series in the parameter ϵ as:

$$\begin{aligned} \phi &= \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \dots, \\ \eta &= \eta_0 + \epsilon \eta_1 + \epsilon^2 \eta_2 + \dots. \end{aligned} \quad (23)$$

Introducing the expansion (23) into the Eqs. (19)–(22) and setting the coefficients of like powers of ϵ equal to zero, the following sets of differential equations are obtained:

$O(1)$ equations:

$$\frac{\partial^2 \phi_0}{\partial z^2} = 0. \quad (24)$$

and the boundary conditions

$$\begin{aligned} \frac{\partial \phi_0}{\partial z} &= 0 \quad \text{at } z = -1, \quad \frac{\partial \phi_0}{\partial z} \Big|_{z=0} = 0, \\ \left[-\frac{\partial \phi_0}{\partial \xi} + \frac{1}{2} \left(\frac{\partial \phi_0}{\partial z} \right)^2 \right] \Big|_{z=0} + \eta_0 &= 0. \end{aligned} \quad (25)$$

$O(\epsilon)$ equations:

$$\frac{\partial^2 \phi_1}{\partial z^2} + \frac{\partial^2 \phi_0}{\partial \xi^2} = 0, \quad (26)$$

and the boundary conditions

$$\begin{aligned} \frac{\partial \phi_1}{\partial z} &= 0, \quad \text{at } z = -1, \quad \frac{\partial \phi_1}{\partial z} \Big|_{z=0} + \frac{\partial \eta_0}{\partial \xi} = 0, \\ \left[-\frac{\partial \phi_1}{\partial \xi} + \frac{\partial \phi_0}{\partial \tau} + \frac{1}{2} \left(\frac{\partial \phi_0}{\partial \xi} \right)^2 + \frac{\partial \phi_0}{\partial z} \frac{\partial \phi_1}{\partial z} \right] \Big|_{z=0} + \eta_1 &= 0. \end{aligned} \quad (27)$$

$O(\epsilon^2)$ equations:

$$\frac{\partial^2 \phi_2}{\partial z^2} + \frac{\partial^2 \phi_1}{\partial \xi^2} = 0, \quad (28)$$

and the boundary conditions

$$\begin{aligned} \frac{\partial \phi_2}{\partial z} &= 0, \quad \text{at } z = -1, \\ \left[\frac{\partial \phi_2}{\partial z} + \eta_0 \frac{\partial^2 \phi_1}{\partial z^2} \right] \Big|_{z=0} + \frac{\partial \eta_1}{\partial \xi} - \frac{\partial \eta_0}{\partial \tau} - \frac{\partial \phi_0}{\partial \xi} \Big|_{z=0} \frac{\partial \eta_0}{\partial \xi} &= 0, \\ \left[\frac{\partial \phi_2}{\partial \xi} + \eta_0 \frac{\partial^2 \phi_1}{\partial z \partial \xi} + \frac{dP_1}{d\tau} \frac{\partial \phi_0}{\partial \xi} - \frac{\partial \phi_1}{\partial \tau} - \frac{1}{2} \left(\frac{\partial \phi_1}{\partial z} \right)^2 \right. \\ \left. - \frac{\partial \phi_0}{\partial \xi} \frac{\partial \phi_1}{\partial \xi} \right] \Big|_{z=0} - \eta_2 &= 0. \end{aligned} \quad (29)$$

$O(\epsilon^3)$ equations:

$$\frac{\partial^2 \phi_3}{\partial z^2} + \frac{\partial^2 \phi_2}{\partial \xi^2} = 0, \quad (30)$$

and the boundary conditions:

$$\begin{aligned} \frac{\partial \phi_3}{\partial z} &= 0, \quad \text{at } z = -1, \\ \left[\frac{\partial \phi_3}{\partial z} + \eta_0 \frac{\partial^2 \phi_2}{\partial z^2} + \eta_1 \frac{\partial^2 \phi_1}{\partial z^2} \right] \Big|_{z=0} + \frac{\partial \eta_2}{\partial \xi} + \frac{dP_1}{d\tau} \frac{\partial \eta_0}{\partial \xi} - \frac{\partial \eta_1}{\partial \tau} \\ - \frac{\partial \phi_0}{\partial \xi} \Big|_{z=0} \frac{\partial \eta_1}{\partial \xi} - \frac{\partial \phi_1}{\partial \xi} \Big|_{z=0} \frac{\partial \eta_0}{\partial \xi} &= 0, \\ \left[\frac{\partial \phi_3}{\partial \xi} + \eta_0 \frac{\partial^2 \phi_2}{\partial \xi \partial z} + \eta_1 \frac{\partial^2 \phi_1}{\partial \xi \partial z} + \frac{1}{2} \eta_0^2 \frac{\partial^3 \phi_1}{\partial z^2 \partial \xi} + \frac{dP_2}{d\tau} \frac{\partial \phi_0}{\partial \xi} \right. \\ + \frac{dP_1}{d\tau} \frac{\partial \phi_1}{\partial \xi} + \frac{\partial \phi_2}{\partial \tau} + \eta_0 \frac{\partial^2 \phi_1}{\partial z \partial \tau} + \frac{1}{2} \left(\frac{\partial \phi_1}{\partial \xi} \right)^2 \\ + \frac{\partial \phi_0}{\partial \xi} \left(\frac{\partial \phi_2}{\partial \xi} + \eta_0 \frac{\partial^2 \phi_1}{\partial z \partial \xi} \right) \\ \left. + \frac{\partial \phi_1}{\partial z} \left(\frac{\partial \phi_2}{\partial z} + \eta_0 \frac{\partial^2 \phi_1}{\partial z^2} \right) \right] \Big|_{z=0} + \eta_3 &= 0. \end{aligned} \quad (31)$$

2.1 Solution of the field equations

From the solution of the sets (24) and (25) one obtains

$$\phi_0 = \varphi_0(\xi, \tau), \quad \eta_0 = \frac{\partial \varphi_0}{\partial \xi}, \quad (32)$$

where $\varphi_0(\xi, \tau)$ is an unknown function of its argument whose evolution equation will be obtained later.

Introducing Eq. (32) into Eqs. (26) and (27), the solution of $O(\epsilon)$ equations gives the following result

$$\begin{aligned} \phi_1 &= -\frac{1}{2} \frac{\partial^2 \varphi_0}{\partial \xi^2} (z^2 + 2z) + \varphi_1(\xi, \tau), \\ \eta_1 &= \frac{\partial \varphi_1}{\partial \xi} - \frac{\partial \varphi_0}{\partial \tau} - \frac{1}{2} \left(\frac{\partial \varphi_0}{\partial \xi} \right)^2, \end{aligned} \quad (33)$$

where $\varphi_1(\xi, \tau)$ is another unknown function to be determined from the higher order perturbation expansion.

Introducing Eqs. (32) and (33) into Eqs. (28) and (29), the solution of $O(\epsilon^2)$ equations may be obtained as

$$\begin{aligned}\phi_2 &= \frac{1}{24} \frac{\partial^4 \varphi_0}{\partial \xi^4} (z^4 + 4z^3 - 8z) - \frac{1}{2} \frac{\partial^2 \varphi_1}{\partial \xi^2} (z^2 + 2z) + \varphi_2(\xi, \tau), \\ \eta_2 &= \frac{\partial \varphi_2}{\partial \xi} - \frac{\partial \varphi_0}{\partial \xi} \frac{\partial^3 \varphi_0}{\partial \xi^3} - \frac{1}{2} \left(\frac{\partial^2 \varphi_0}{\partial \xi^2} \right)^2 \\ &\quad - \frac{\partial \varphi_0}{\partial \xi} \frac{\partial \varphi_1}{\partial \xi} + \lambda_0 \frac{\varphi_0}{\partial \xi} - \frac{\partial \varphi_1}{\partial \tau},\end{aligned}\quad (34)$$

where $\varphi_2(\xi, \tau)$ is another unknown function whose evolution equation will be obtained from the higher order perturbation expansion. The use of the second boundary condition in (33) yields

$$\frac{\partial^2 \varphi_0}{\partial \xi \partial \tau} + \frac{3}{2} \frac{\partial \varphi_0}{\partial \xi} \frac{\partial^2 \varphi_0}{\partial \xi^2} + \frac{1}{6} \frac{\partial^4 \varphi_0}{\partial \xi^4} = 0. \quad (35)$$

Noting the relation $\partial \varphi_0 / \partial \xi = \eta_0$, Eq. (35) reduces to the following Korteweg-de Vries (KdV) equation

$$\frac{\partial \eta_0}{\partial \tau} + \frac{3}{2} \eta_0 \frac{\partial \eta_0}{\partial \xi} + \frac{1}{6} \frac{\partial^3 \eta_0}{\partial \xi^3} = 0. \quad (36)$$

To obtain the solution for $O(\epsilon^3)$ equations we introduce Eqs. (32), (33) and (34) into Eqs. (30) and (31), which results in

$$\begin{aligned}\frac{\partial^2 \phi_3}{\partial z^2} + \frac{1}{24} \frac{\partial^6 \varphi_0}{\partial \xi^6} (z^4 + 4z^3 - 8z) \\ - \frac{1}{2} \frac{\partial^4 \varphi_1}{\partial \xi^4} (z^2 + 2z) + \frac{\partial^2 \varphi_2}{\partial \xi^2} = 0,\end{aligned}\quad (37)$$

and the boundary conditions

$$\begin{aligned}\frac{\partial \phi_3}{\partial z} \Big|_{z=0} - \eta_0 \frac{\partial^2 \varphi_1}{\partial \xi^2} - \eta_1 \frac{\partial^2 \varphi_0}{\partial \xi^2} + \frac{\partial \eta_2}{\partial \xi} + \frac{dP_1}{d\tau} \frac{\partial \eta_0}{\partial \xi} - \frac{\partial \eta_1}{\partial \tau} \\ - \frac{\partial \varphi_0}{\partial \xi} \frac{\partial \eta_1}{\partial \xi} - \frac{\partial \varphi_1}{\partial \xi} \frac{\partial \eta_0}{\partial \xi} = 0.\end{aligned}\quad (38)$$

The solution of (37) after the use of the first boundary condition in (38) gives

$$\begin{aligned}\phi_3 &= -\frac{1}{720} \frac{\partial^6 \varphi_0}{\partial \xi^6} (z^6 + 6z^5 - 40z^3 + 96z) \\ &\quad + \frac{1}{24} \frac{\partial^4 \varphi_1}{\partial \xi^4} (z^4 + 4z^3 - 8z) \\ &\quad - \frac{1}{2} \left(\frac{\partial^2 \varphi_2}{\partial \xi^2} \right) (z^2 + 2z) + \varphi_3(\xi, \tau),\end{aligned}\quad (39)$$

where $\varphi_3(\xi, \tau)$ is another unknown function to be determined from the higher order perturbation expansion. Employing the last boundary condition in (38) the following evolution equation is obtained

$$\frac{\partial \eta_1}{\partial \tau} + \frac{3}{2} \frac{\partial}{\partial \xi} (\eta_0 \eta_1) + \frac{1}{6} \frac{\partial^3 \eta_1}{\partial \xi^3} = S(\eta_0), \quad (40)$$

where the function $S(\eta_0)$ is defined by

$$\begin{aligned}S(\eta_0) &= \frac{\partial}{\partial \xi} \left[-\frac{19}{360} \frac{\partial^4 \eta_0}{\partial \xi^4} - \frac{13}{48} \left(\frac{\partial \eta_0}{\partial \xi} \right)^2 \right. \\ &\quad \left. - \frac{5}{12} \frac{\partial}{\partial \xi} \eta_0 \frac{\partial^2 \eta_0}{\partial \xi^2} + \frac{\eta_0^3}{8} + \frac{dP_1}{d\tau} \eta_0 \right].\end{aligned}\quad (41)$$

Eqs. (40) and (41) are the same as those proposed by Kodama and Taniuti [6], who named it as the “re-normalization” method. Here we note that the zeroth order solution in the perturbation expansion plays the role a source to generate the first order solution.

2.2 Solitary waves

In this sub-section we shall study the localized travelling wave solution to the evolution equations (36) and (40). For that purpose we introduce

$$\eta_i = \eta_i(\zeta, \tau), \quad \zeta = \alpha(\xi - ct), \quad (i = 0, 1), \quad (42)$$

where the constants α and c are to be determined from the solution.

Introducing Eq. (42) for $i = 0$ into the evolution equation (36) we have

$$-c\eta'_0 + \frac{3}{2} \eta_0 \eta'_0 + \frac{\alpha^2}{6} \eta_0''' = 0, \quad (43)$$

where the prime denotes the differentiation of the corresponding quantity with respect to ζ . Integrating Eq. (43) with respect to ζ and utilizing the localization condition, i.e., η_0 and its various order derivatives vanish as $\zeta \rightarrow \pm\infty$, we obtain

$$-c\eta_0 + \frac{3}{4} \eta_0^2 + \frac{\alpha^2}{6} \eta_0'' = 0. \quad (44)$$

Eq. (44) admits the solitary wave solution of the form

$$\eta_0 = a \operatorname{sech}^2 \zeta, \quad (45)$$

where a is the amplitude of the solitary wave. Inserting Eq. (45) into (44) and setting the coefficients of various powers of $\text{sech } \zeta$ equal to zero we obtain

$$\alpha = \left(\frac{3a}{4}\right)^{1/2}, \quad c = \frac{a}{2}. \quad (46)$$

For the solution of the evolution equation (40), introducing Eq. (42) for $i = 1$ into Eqs. (40) and (41), integrating the result with respect to ζ and utilizing the localization condition we obtain

$$-\frac{a}{2}\eta_1 + \frac{3}{2}(\eta_0\eta_1) + \frac{a}{8}\eta_1'' = -\frac{19a^2}{640}\eta_0^{(4)} - \frac{13a}{64}(\eta_0')^2 - \frac{5a}{16}\eta_0\eta_0'' + \frac{\eta_0^3}{8} + \frac{dP_1}{d\tau}\eta_0. \quad (47)$$

Noting the following relations:

$$\begin{aligned} \eta_0'' &= 4\eta_0 - \frac{6}{a}\eta_0^2, \quad (\eta_0')^2 = 4\eta_0^2 - \frac{4}{a}\eta_0^3, \\ \eta_0^{(4)} &= 16\eta_0 - \frac{120}{a}\eta_0^2 + \frac{120}{a^2}\eta_0^3, \end{aligned} \quad (48)$$

Eq. (47) becomes

$$\eta_1'' + \left(\frac{12}{a}\eta_0 - 4\right)\eta_1 = \left(\frac{8}{a}\frac{dP_1}{d\tau} - \frac{19a}{5}\right)\eta_0 + 12\eta_0^2 - \frac{6}{a}\eta_0^3. \quad (49)$$

Since η_0' is one of the fundamental solutions of the homogeneous equation in (49), the first term on the right hand side causes to secularity in the solution of η_1 . In order to remove the secularity the coefficient of η_0 must vanish.

$$\frac{8}{a}\frac{dP_1}{d\tau} - \frac{19a}{5} = 0, \quad \text{or} \quad P_1(\tau) = \frac{19a^2}{40}\tau, \quad (50)$$

and the remaining part of Eq. (49) becomes

$$\eta_1'' + \left(\frac{12}{a}\eta_0 - 4\right)\eta_1 = 12\eta_0^2 - \frac{6}{a}\eta_0^3. \quad (51)$$

The particular solution of Eq. (51) gives

$$\eta_1 = \frac{a}{2}\eta_0 + \frac{3}{4}\eta_0^2. \quad (52)$$

Thus, in terms of real physical quantities, the total solution up to $O(\epsilon^2)$ may be given by

$$\eta = a \text{sech}^2 \zeta + \epsilon a^2 \left(\frac{1}{2} \text{sech}^2 \zeta + \frac{3}{4} \text{sech}^4 \zeta \right), \quad (53)$$

where

$$\zeta = \left(\frac{3a}{4}\right)^{1/2} \epsilon^{1/2} \left[x - \left(1 + \epsilon \frac{a}{2} + \epsilon^2 \frac{19a^2}{40} + \dots \right) t \right]. \quad (54)$$

This solution is exactly the same with those of obtained by Demiray [12] who employed the modified reductive perturbation method. The contribution of higher order terms to the solitary wave solution may be obtained in a like manner, but the calculations become rather involved. Here it is to be noted that as we add more terms in the perturbation expansion the expressions of ξ and ζ will be changed.

3 Conclusion

Employing the modified PLK method, the propagation of weakly nonlinear waves in a shallow water is studied and a set of KdV equations are obtained as the evolution equations. These evolution equations are exactly the same with those of proposed by Kodama and Taniuti [6], in which they employed the so-called “renormalization method”, which was introduced in an ad hoc manner. In this work we proved the validity of renormalization method through the use of modified PLK method. By seeking a progressive wave solution to these evolution equations the speed correction terms are obtained so as to remove possible secularities in the solution. It is observed that the lowest order wave speed corresponds to the wave speed found in conventional reductive perturbation method, whereas the higher order terms in the perturbation expansion lead to the speed correction terms. Finally, we should point out that the present result is consistent with the results of Demiray [12], in which the modified reductive perturbation method was utilized.

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